Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 42 (2005) 455–476
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Effective elastic properties of matrix composites with transversely-isotropic phases

Igor Sevostianov [a,*], Nadir Yilmaz [a], Vladimir Kushch [b], Valery Levin [c]

[a] *Department of Mechanical Engineering, New Mexico State University, P.O. Box 30001, Las Cruces, NM 88003, USA*
[b] *Institute for Superhard Materials, Ukraine National Academy of Science, 254074 Kiev, Ukraine*
[c] *Institute Mexicano Del Petroleo, C.P. 07730, Mexico D.F., Mexico*

Received 9 April 2004; received in revised form 14 June 2004
Available online 25 August 2004

Abstract

The present work addresses the problem of calculation of the macroscopic effective elastic properties of composites containing transversely isotropic phases. As a first step, the contribution of a single inhomogeneity to the effective elastic properties is quantified. Relevant stiffness and compliance contribution tensors are derived for spheroidal inhomogeneities. The limiting cases of spherical, penny-shaped and cylindrical shapes are discussed in detail. The property contribution tensors are used to derive the effective elastic moduli of composite materials formed by transversely isotropic phases in two approximations: non-interaction approximation and effective field method. The results are compared with elastic moduli of quasi-random composites.

Keywords: Composites; Effective properties; Eshelby tensor; Transversely isotropic phases

1. Introduction

Evaluation of the effective elastic properties of inhomogeneous materials has a very long history but it is still one of the most actual problems of micromechanics. In a contrast with the composites containing isotropic phases, very few explicit analytical results can be found in literature related to three-dimensional matrix composites with *anisotropic* components. It is related to significant mathematical difficulties appearing in such problems.

* Corresponding author. Tel.: +1 505 6463322; fax: +1 505 6466111.
E-mail address: igor@me.nmsu.edu (I. Sevostianov).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.06.047

We should note that the flux of papers on the effective elastic properties of composites with isotropic constituents was inspired by the celebrated papers of Eshelby (1957, 1959, 1961) about a single ellipsoidal inhomogeneity. The review of methods developed on this background was given by Hashin (1983).

Mura (1982) has derived an integral form of the Eshelby's type solution for a single inclusion in an anisotropic medium. However, the implicit character of the solution does not allow one to apply it directly to the calculation of the effective properties of composites. Seven years later, Withers (1989) obtained components of the Eshelby's tensor for a spheroidal inhomogeneity embedded in a transversely isotropic matrix. Recently, Kushch and Sevostianov (2004) derived expression for the effective elastic stiffness tensor of a transversely isotropic elastic solid containing arbitrarily placed spherical inclusions, employing the method of multipole expansion for series solution. We also have to mention several papers addressing composite materials with transversely isotropic *piezoelectric* phases. Elastic properties can be obtained from these results as a limiting case. The effective properties of piezocomposites are derived in explicit form for fiber reinforced materials. Several methods of averaging were proposed for this aim. Comparison of the various schemes and detailed literature review is given by Sevostianov et al. (2001).

The present paper constitutes a further progress in these studies. A unified description covering inhomogeneities of diverse shapes is developed. The approach yields in a unified way, the effective elastic moduli for an inhomogeneous material consisting of transversely isotropic phases. The effective moduli are derived in two approximations:

(a) Non-interaction approximation. This approximation appears to be the most important one, since it serves as a basis for various one particle approximations that account for interaction by placing non-interacting inhomogeneities into some "effective environment" (either effective matrix, or effective elastic field).
(b) Effective field method proposed by Kanaun (1983), Kanaun and Levin (1993, 1994), Markov (1999). In this method, the interaction between inhomogeneities is accounted for by placing a representative inclusion into the average stress (or strain) field.

The analysis is done in the framework of linear elasticity and covers the case when axes of geometrical symmetry of spheroidal inhomogeneities coincide with the axes of the material symmetry of the matrix. Axes of anisotropy of the inhomogeneities are not necessarily aligned with them. In the Cartesian coordinate system, $Oxyz$, with Oz axis aligned with the anisotropy axis of transversely isotropic elastic material, the generalized Hook's law has the following form:

$$\begin{aligned}
\sigma_{11} &= C^0_{1111}\varepsilon_{11} + C^0_{1122}\varepsilon_{22} + C^0_{1133}\varepsilon_{33}, \quad \sigma_{13} = 2C^0_{1313}\varepsilon_{13} \\
\sigma_{22} &= C^0_{1122}\varepsilon_{11} + C^0_{2222}\varepsilon_{22} + C^0_{2233}\varepsilon_{33}, \quad \sigma_{23} = 2C^0_{1313}\varepsilon_{23} \\
\sigma_{33} &= C^0_{1133}\varepsilon_{11} + C^0_{2233}\varepsilon_{22} + C^0_{3333}\varepsilon_{33}, \quad \sigma_{12} = \left(C^0_{1111} - C^0_{1122}\right)\varepsilon_{12}
\end{aligned} \tag{1.1}$$

where C^0_{ijkl} are components of the elastic stiffness tensor. The components of the stress tensor $\boldsymbol{\sigma}_{ij}$ satisfy the elastic equilibrium equations $\boldsymbol{\sigma}_{ij,j} = 0$ and the strain tensor $\boldsymbol{\varepsilon}_{ij}$ is related to the displacement vector $\mathbf{u}_i$ by $\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}(\mathbf{u}_{i,j} + \mathbf{u}_{j,i})$.

Our analysis follows methodology developed by Kachanov et al. (1994) and is based on one-particle solution for transversely isotropic material. First, we obtain expressions for stiffness and compliance contribution tensors of a spheroidal inhomogeneity embedded in a transversely isotropic matrix. The results are specified for three geometries of inhomogeneities—strongly oblate, spherical, and strongly prolate ones. The case of isotropic matrix (see Sevostianov and Kachanov, 2002) is recovered as a limiting case. Then, the stiffness and compliance contribution tensors are used to calculate the effective elastic properties of the composite.

2. Single inhomogeneity

2.1. Statement of the problem

We consider a certain reference volume V of an infinite three-dimensional transversely isotropic medium with an embedded inclusion of volume V_*—a region possessing elastic properties different from those of the surrounding material. The properties of the inclusion and of the matrix will be denoted by an asterisk and by "0", respectively. We assume the perfect contact between the matrix and the inclusion:

$$[u_i] = 0, \quad [\sigma_{ij} n_j] = 0 \tag{2.1}$$

where u_i is displacement vector, σ_{ij} is the stress tensor and n_j is unit vector normal to the interface. The bracket stands for the difference between the values of a function at different sides of the interface.

We start from the system of equilibrium and compatibility equations for the medium with a single inclusion

$$\mathrm{div}[\sigma_{kl}(x)] = 0, \quad \mathrm{Curl}[S_{ijkl}(x)\sigma_{kl}(x)] = 0 \tag{2.2}$$

where the Curl is the operator of compatibility. We assume that the tensor $S_{ijkl}(x)$ is represented in the form

$$S_{ijkl}(x) = S^0_{ijkl} + S^1_{ijkl} V(x), \quad S^1_{ijkl} = S^*_{ijkl} - S^0_{ijkl} \tag{2.3}$$

where $V(x)$ is the characteristic function of the domain V_*. This allows to write

$$\mathrm{div}\,\sigma_{kl}(x) = 0, \quad \mathrm{Curl}[S^0_{ijkl}\sigma_{kl}(x)] = -\mathrm{Curl}[S^1_{ijkl}\sigma_{kl}(x)]V(x) \tag{2.4}$$

Using an ordinary procedure (see Kunin, 1983) the system of differential equations can be replaced by equivalent integral equation as follows:

$$\sigma_{kl}(x) = \sigma^0_{kl}(x) + \int_V Q_{klij}(x - x')[S^1_{ijkl}\sigma_{kl}(x')]\mathrm{d}x' \tag{2.5}$$

where $\sigma^0_{kl}(x)$ is the external field which satisfies the equation

$$\mathrm{Curl}[S^0_{ijkl}\sigma^0_{kl}(x)] = 0 \tag{2.6}$$

and given conditions at infinity. In other words, $\sigma^0_{kl}(x)$ is the solution of corresponding homogeneous equation. The kernel of Eq. (2.5) can be expressed via the second derivatives of the Green tensor G_{ij} for displacement in elastic medium as follows:

$$\begin{aligned} &Q_{ijmn}(x) = -C^0_{ijpq}[I_{pqmn}\delta(x) + P_{pqkl}(x)C^0_{klmn}], \quad P_{ijkl}(x) = G_{i)(k,l)(j}(x) \\ &C^0_{klmn} = (S^0_{klmn})^{-1}, \quad I_{ijmn} = \delta_{i(m}\delta_{n)j} \end{aligned} \tag{2.7}$$

where the parenthesis in subscripts means operation of symmetrization by corresponding indices, for example $\delta_{i(m}\delta_{n)j} = \frac{1}{2}(\delta_{im}\delta_{nj} + \delta_{in}\delta_{mj})$.

Let us consider the Cartesian coordinate system with axes coinciding with the semi-axes a_i of the ellipsoidal domain V_*. Then, the ellipsoid can be described using the following relation:

$$x_k(a_{kj})x_j \leqslant 1, \quad a_{ij} = a_i^{-2}\delta_{ij} \tag{2.8}$$

If the external field $\sigma^0_{ij}(x)$ is uniform in the domain V, then the stress field $\sigma_{ij}(x)$ is also uniform inside V (Eshelby, 1957) and can be determined by the following relations:

$$\begin{aligned}\sigma_{ij}(x) &= [I_{ijkl} + Q_{ijmn}(a)S^1_{mnkl}]^{-1}[\sigma^0_{kl}(x)], \quad x \in V \\ Q_{ijmn}(a) &= C^0_{ijpq}[I_{pqmn} + P_{pqrs}(a)C^0_{rsmn}] \\ P_{pqrs}(a_{ij}) &= \frac{1}{4\pi}\int_{\Omega_1} P^*_{pqrs}(a^{-1}_{ij}k)\mathrm{d}\Omega\end{aligned} \tag{2.9}$$

where the $P^*_{pqrs}(k)$ is the Fourier transform of the tensor function $P_{pqrs}(x)$ and the integration is carried out over the unit sphere Ω_1. If the main medium (matrix) is isotropic, tensor $P_{pqrs}(a)$ has the ellipsoidal symmetry and is defined by nine essential components which can be expressed in terms of elliptical integrals. It is reduced to elementary functions in the case of spheroidal shape. The explicit relations for these tensors in the case of isotropic linear elastic medium are well known (see Sevostianov and Kachanov, 2002, for example). In the text to follow, we determine tensor P_{ijkl} for a transversely isotropic matrix.

Our analysis requires explicit analytic inversions of fourth rank tensors. Such inversions can be done by representing these tensors in terms of a certain "standard" tensorial basis $T^{(1)}_{ijkl}, \ldots, T^{(6)}_{ijkl}$ (Kunin, 1983; see Appendix A):

$$P_{ijkl} = \sum_{k=1}^{6} p_k T^{(k)}_{ijkl}, \quad Q_{ijkl} = \sum_{k=1}^{6} q_k T^{(k)}_{ijkl} \tag{2.10}$$

2.2. Single inhomogeneity in a transversely isotropic material

At the first step, we will derive explicit expression for tensor P_{ijkl} in the case of transversely isotropic matrix

$$P_{ijkl} = \int_V G_{ik,lj}(x - x')\mathrm{d}x'|_{(ij)(kl)} \tag{2.11}$$

where $G(x)$ is the Green's function for the anisotropic unbounded medium and the symbol parenthesis ( ) stands for the symmetrization over corresponding indices. In the arbitrary anisotropic medium, the Green's function can be represented in the form

$$G_{ik}(x) = \frac{1}{r}\Gamma_{ik}(\theta, \varphi) \tag{2.12}$$

where (r, θ, φ) is the spherical coordinate system. Applying the approach developed by Vakulenko (1998), Eq. (2.14) can be transformed in the surface integral over the unit sphere Ω.

$$\mathbf{P} = \mathbf{E} \cdot \int_\Omega (\mathbf{e}^r \cdot \mathbf{E} \cdot \mathbf{e}^r)^{-1}\mathbf{e}^r[\nabla^*\boldsymbol{\Gamma}(\mathbf{e}^r) - \mathbf{e}^r\boldsymbol{\Gamma}(\mathbf{e}^r)]\mathrm{d}\Omega \tag{2.13}$$

where $\mathbf{e}^r$, $\mathbf{e}^\theta$, $\mathbf{e}^\varphi$ are the basis vector of spherical coordinate system and

$$\nabla^* = \frac{\mathbf{e}^\varphi}{\sin\theta} \cdot \frac{\partial}{\partial\varphi} + \mathbf{e}^\theta \cdot \frac{\partial}{\partial\theta} \tag{2.14}$$

In Eq. (2.16), the second rank tensor $\mathbf{E}$ depends on the inclusion form and for the spheroidal inclusion $(a_1 = a_2 = a, a_3)$ it is defined by the following expression:

$$E_{ij} = \frac{1}{a^2}(\theta_{ij} + \xi^2 m_i m_j), \quad \xi = \frac{a}{a_3}, \quad \theta_{ij} = \delta_{ij} - m_i m_j \tag{2.15}$$

where m_i is the unit vector in the z-axis. Taking into account that the medium is transversely isotropic, it is convenient to find the integral (2.16) in the system of cylindrical coordinates. For this purpose, tensor $\boldsymbol{\Gamma}(\theta)$ and $\mathbf{E}$ should be rewritten as the function of (ρ, φ, z) coordinates. Tensor $\boldsymbol{\Gamma}(\theta)$ and its components for the elastic field in transversely isotropic medium have been obtained explicitly by Kroner (1953) (and result was corrected by Yoo, 1974). Thus,

$$\Gamma_{ik}(\theta,\varphi) = \Gamma_{\varphi\varphi}(\theta)e_i^{\varphi}e_k^{\varphi} + \Gamma_{\rho\rho}(\theta)e_i^{\rho}e_k^{\rho} + \Gamma_{\rho z}(\theta)(e_i^{\rho}e_k^{z} + e_k^{\rho}e_i^{z}) + \Gamma_{zz}(\theta)e_i^{z}e_k^{z} \tag{2.16}$$

where the quantities $\Gamma_{\varphi\varphi}(\theta)$, $\Gamma_{\rho\rho}(\theta)$, $\Gamma_{\rho z}(\theta)$ and $\Gamma_{zz}(\theta)$ are expressed as

$$\begin{aligned}
\Gamma_{\varphi\varphi}(\theta) &= \sum_{l=1}^{3} \frac{(b_l - a_l A_l)\sin^2\theta - a_l\cos^2\theta}{\sin^2\theta\sqrt{A_l\sin^2\theta + \cos^2\theta}}\\
\Gamma_{\rho\rho}(\theta) &= \sum_{l=1}^{3} \frac{b_l\sin^2\theta + a_l\cos^2\theta}{\sin^2\theta\sqrt{A_l\sin^2\theta + \cos^2\theta}}\\
\Gamma_{\rho z}(\theta) &= \sum_{l=1}^{3} \frac{c_l\cos\theta}{\sin\theta\sqrt{A_l\sin^2\theta + \cos^2\theta}}\\
\Gamma_{zz}(\theta) &= \sum_{l=1}^{3} \frac{d_l}{\sqrt{A_l\sin^2\theta + \cos^2\theta}}
\end{aligned} \tag{2.17}$$

where coefficients a_l, b_l, c_l, and d_l and A_1, A_2, and A_3 depend on the components of the tensor of elastic moduli and these coefficients can be represented as

$$\begin{aligned}
a_l &= \frac{1}{\varepsilon_l}\left[(C_{1212} - C_{1111})(C_{3333} - A_l C_{2323}) + (C_{1133} + C_{2323})^2\right]\\
b_l &= \frac{1}{\varepsilon_l}\left[(C_{2323} - A_l C_{1111})(C_{3333} - A_l C_{2323}) + A_l(C_{1133} + C_{2323})^2\right]\\
c_l &= \frac{1}{\varepsilon_l}(C_{1133} + C_{2323})(C_{2323} - A_l C_{1212})\\
d_l &= \frac{1}{\varepsilon_l}(C_{2323} - A_l C_{1111})(C_{2323} - A_l C_{1212})\\
\varepsilon_l &= 4\pi C_{1111}C_{2323}C_{1212}\prod_{\substack{j=1\\(j\neq l)}}^{3}(A_j - A_l)\\
A_1 &= \frac{C_{2323}}{C_{1212}}
\end{aligned} \tag{2.18}$$

where A_2 and A_3 are the roots of the quadratic equation

$$C_{1111}C_{2323}A^2 + ((C_{1133})^2 + 2C_{1133}C_{2323} - C_{1111}C_{3333})A + C_{3333}C_{2323} = 0 \tag{2.19}$$

In Eq. (2.19), the basis vectors of cylindrical coordinates system $\mathbf{e}^{\rho}, \mathbf{e}^{\varphi}, \mathbf{e}^{z}$ are

$$\mathbf{e}^{\rho} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix}, \quad \mathbf{e}^{\varphi} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \quad \mathbf{e}^{z} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \tag{2.20}$$

Using the relationships between the basis vectors for spherical and cylindrical coordinate systems

$$\mathbf{e}^{r} = \mathbf{e}^{\rho}\sin\theta + \mathbf{e}^{z}\cos\theta, \quad \mathbf{e}^{\theta} = \mathbf{e}^{\rho}\cos\theta - \mathbf{e}^{z}\sin\theta \tag{2.21}$$

one can rewrite operator ∇^* in the form

$$\nabla^* = \frac{\mathbf{e}^\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} + (\mathbf{e}^\rho\cos\theta - \mathbf{e}^z\sin\theta)\frac{\partial}{\partial\theta} \tag{2.22}$$

Taking into account that for the spheroidal inclusion

$$\mathbf{e}^r\cdot\mathbf{E}\cdot\mathbf{e}^r = \frac{1}{a^2}(\sin^2\theta + \xi^2\cos^2\theta) \tag{2.23}$$

$$\begin{aligned}
&\int_0^{2\pi} e_i^\varphi e_j^\varphi \mathrm{d}\varphi = \int_0^{2\pi} e_i^\rho e_j^\rho \mathrm{d}\varphi = \pi\theta_{ij}\\
&\int_0^{2\pi} e_i^\rho e_j^\rho e_k^\rho e_l^\rho \mathrm{d}\varphi = \frac{\pi}{4}(\theta_{ij}\theta_{kl} + \theta_{ik}\theta_{lj} + \theta_{il}\theta_{kj})\\
&\int_0^{2\pi} e_i^\rho e_j^\rho e_k^\varphi e_l^\varphi \mathrm{d}\varphi = \frac{\pi}{4}(3\theta_{ij}\theta_{kl} - \theta_{ik}\theta_{lj} - \theta_{il}\theta_{kj})
\end{aligned} \tag{2.24}$$

we obtain after φ-integration and applying the T-basis (Appendix A)

$$P_{ijkl} = \sum_{l=1}^{3}\int_0^\pi P_{ijkl}^{(l)}(\theta)\sin\theta\mathrm{d}\theta \tag{2.25}$$

where

$$\begin{aligned}
P_{ijkl}^{(l)}(\theta) = -\frac{\pi}{2\Delta_l}\{&(b_l - A_l a_l)A_l\sin^2\theta T_{ijkl}^1 + (2b_l - A_l a_l)A_l\sin^2\theta T_{ijkl}^2 - c_l A_l(\sin^2\theta - \xi^2\cos^2\theta)(T_{ijkl}^3 + T_{ijkl}^4)\\
&+ \left[2\xi^2(2b_l - A_l a_l)\cos^2\theta - 2c_l A_l(\sin^2\theta - \xi^2\cos^2\theta) + 2d_l A_l\sin^2\theta\right]T_{ijkl}^5 + 4d_l\xi^2\cos^2\theta T_{ijkl}^6\}
\end{aligned} \tag{2.26}$$

$$\Delta_l = (A_l\sin^2\theta + \cos^2\theta)^{3/2}(\sin^2\theta + \xi^2\cos^2\theta) \tag{2.27}$$

Finally, the integration in (2.28) over the angle θ leads to

$$P_{ijkl} = P_1 T_{ijkl}^1 + P_2 T_{ijkl}^2 + P_3 T_{ijkl}^3 + P_4 T_{ijkl}^4 + P_5 T_{ijkl}^5 + P_6 T_{ijkl}^6 \tag{2.28}$$

The coefficients P_1, P_2, P_3, P_4, P_5 and P_6 are obtained as follows:

$$\begin{aligned}
&P_1 = -\frac{\pi}{2}\sum_{l=1}^{3}(b_l - A_l a_l)J_1^{(l)}, \quad P_2 = -\frac{\pi}{2}\sum_{l=1}^{3}(2b_l - A_l a_l)J_1^{(l)},\\
&P_3 = \frac{\pi}{2}\sum_{l=1}^{3}c_l(J_1^{(l)} - \xi^2 A_l J_2^{(l)}), \quad P_4 = \frac{\pi}{2}\sum_{l=1}^{3}c_l(J_1^{(l)} - \xi^2 A_l J_2^{(l)})\\
&P_5 = -\pi\sum_{l=1}^{3}\left[(2b_l - A_l a_l)\xi^2 J_2^{(l)} - c_l(J_1^{(l)} - \xi^2 A_l J_2^{(l)}) + d_l J_1^{(l)}\right],\\
&P_6 = -2\pi\sum_{l=1}^{3}d_l\xi^2 J_2^{(l)}
\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}
&J_1^{(l)} = A_l\int_{-1}^{1}\frac{(1-u^2)\mathrm{d}u}{[1+(\xi^2-1)u^2][A_l+(1-A_l)u^2]^{3/2}} = 2\lambda_l^2\left[1 - \xi^2 A_l\lambda_l\ln\left(\frac{\lambda_l+1}{\lambda_l-1}\right)\right]\\
&J_2^{(l)} = \int_{-1}^{1}\frac{u^2\mathrm{d}u}{[1+(\xi^2-1)u^2][A_l+(1-A_l)u^2]^{3/2}} = 2\lambda_l^2\left[\frac{1}{2}\lambda_l\ln\left(\frac{\lambda_l+1}{\lambda_l-1}\right) - 1\right]\\
&\lambda_l = \sqrt{\frac{1}{1-A_l\xi^2}}
\end{aligned} \tag{2.30}$$

Eshelby's tensor is related to tensor P_{ijkl} as follows:

$$S^{\mathrm{E}}_{ijkl} = P_{ijmn}C^0_{mnkl} \tag{2.31}$$

Using the algebra of tensor basis given in Appendix A, we can write tensor S^{E}_{ijkl} in the form

$$S^{\mathrm{E}}_{ijkl} = \sum_{i=1}^{6} S^{\mathrm{E}}_i T^{(i)}_{ijkl} \tag{2.32}$$

with the coefficients S^{E}_i are as follows:

$$\begin{aligned} &S^{\mathrm{E}}_1 = 2P_1C^0_1 + P_3C^0_4, \quad S^{\mathrm{E}}_2 = P_2C^0_2, \quad S^{\mathrm{E}}_3 = 2P_1C^0_3 + P_3C^0_6, \quad S^{\mathrm{E}}_4 = 2P_4C^0_1 + P_6C^0_4, \\ &S^{\mathrm{E}}_5 = \frac{1}{2}P_5C^0_5, \quad S^{\mathrm{E}}_6 = P_6C^0_6 + 2P_4C^0_3 \end{aligned} \tag{2.33}$$

where P_{1-6} are given by (2.32) and C^0_{1-6} are given by (A.8) in Appendix A. Expressions for components of Eshelby's tensor (the connection between Cartesian components and representation in terms of tensor basis is given by (A.7)) completely coincide with the solution of Withers (1989). Fig. 1 illustrates it in the case of the following elastic constants:

$$C^0_{1111} = 2.179, \quad C^0_{1122} = 0.579, \quad C^0_{1133} = 0.689, \quad C^0_{2323} = 1, \quad C^0_{3333} = 10.345 \tag{2.34}$$

To calculate effective elastic properties of inhomogeneous material it is convenient to use property contribution tensors H_{ijkl} and N_{ijkl} (see Sevostianov and Kachanov, 2002; Kachanov et al., 2004)

$$\begin{aligned} H_{ijkl} &= \frac{V_*}{V}[(S^*_{ijkl} - S^0_{ijkl})^{-1} + Q_{ijkl}]^{-1} \\ N_{ijkl} &= \frac{V_*}{V}[(C^*_{ijkl} - C^0_{ijkl})^{-1} - P_{ijkl}]^{-1} \end{aligned} \tag{2.35}$$

Below, we specify our results for three limiting cases of primary interest: $\xi \to \infty$ (strongly oblate inhomogeneity), $\xi \to 0$ (strongly prolate inhomogeneity) and $\xi = 1$ (spherical inhomogeneity).

2.2.1. Strongly oblate spheroidal inhomogeneity

In the case of the strongly oblate (penny-shaped) geometry of the inhomogeneity, $\xi = \frac{a}{a_3} \to \infty$ and the expressions for the shape factors (2.30) are reduced to

$$J^{(l)}_1 = \frac{\pi}{\xi\sqrt{A_l}} + O\left(\frac{1}{\xi^2}\right), \quad \xi^2 J^{(l)}_2 = \frac{2}{A_l}\left(1 - \frac{\pi}{2\xi\sqrt{A_l}}\right) + O\left(\frac{1}{\xi^2}\right) \tag{2.36}$$

Substitution of these expressions into (2.29) gives the following formulas for tensor $\mathbf{P}$ and its coefficients P_{1-6}:

$$\mathbf{P} = \mathbf{P}^0 + \frac{\pi}{\xi}\mathbf{P}^\xi + O\left(\frac{1}{\xi^2}\right) \tag{2.37}$$

$$P_1 = 0, \quad P_2 = 0, \quad P_3 = 0, \quad P_4 = 0, \quad P_5 = \frac{4}{C^0_5}, \quad P_6 = \frac{1}{C^0_6} \tag{2.38}$$

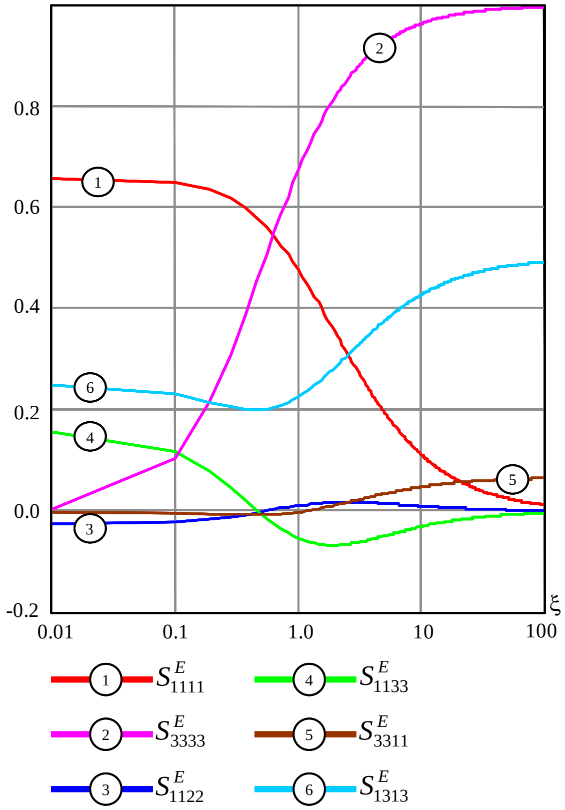

Fig. 1. Dependence of the components of Eshelby's tensor on the aspect ratio of an inhomogeneity. The stiffnesses of the matrix are $C^0_{1111} = 2.179$, $C^0_{1122} = 0.579$, $C^0_{1133} = 0.689$, $C^0_{2323} = 1$, and $C^0_{3333} = 10.345$. The curves are completely coincide with those calculated by formulas of Withers (1989).

$$
\begin{aligned}
P_1^\xi &= \frac{4C_6^0 + C_5^0\sqrt{A_2A_3}}{4C_5^0(2C_1^0 + C_2^0)(\sqrt{A_2}+\sqrt{A_3})\sqrt{A_2A_3}} \\
P_2^\xi &= \frac{1}{2C_5^0}\left[\sqrt{\frac{C_5^0}{2C_2^0}} + \frac{4C_6^0 + C_5^0\sqrt{A_2A_3}}{2(2C_1^0 + C_2^0)(\sqrt{A_2}+\sqrt{A_3})\sqrt{A_2A_3}}\right] \\
P_3^\xi &= -\frac{4C_3^0 + C_5^0}{2C_5^0(2C_1^0 + C_2^0)(\sqrt{A_2}+\sqrt{A_3})\sqrt{A_2A_3}} \\
P_4^\xi &= -\frac{4C_3^0 + C_5^0}{2C_5^0(2C_1^0 + C_2^0)(\sqrt{A_2}+\sqrt{A_3})\sqrt{A_2A_3}} \\
P_5^\xi &= -\frac{1}{C_5^0}\left[\sqrt{\frac{2C_2^0}{C_5^0}} + \frac{4(2C_1^0 + C_2^0)C_6^0 + 8(C_3^0)^2}{(2C_1^0 + C_2^0)C_5^0(\sqrt{A_2}+\sqrt{A_3})\sqrt{A_2A_3}}\right] \\
P_6^\xi &= \frac{-C_5^0(A_2 + A_3 + \sqrt{A_2A_3}) + 4C_6^0}{2C_6^0C_5^0(\sqrt{A_2}+\sqrt{A_3})\sqrt{A_2A_3}}
\end{aligned}
\tag{2.39}
$$

Note that for the computation of the principal term in expansion of $\mathbf{P}^{-1}$ with respect to ξ we have to retain first two terms in $\mathbf{P}$ since tensor $\mathbf{P}^0$ in (2.37) does not have an inverse. Eshelby's tensor can be expressed in terms of basic tensors as follows:

$$S^{\mathrm{E}}_{\text{Penny-Shaped}} = \frac{C^0_4}{C^0_6} T^{(4)} + 2T^{(5)} + T^{(6)} \tag{2.40}$$

In the case of rigid disk of radius a embedded into an elastic material (of volume V), we can express tensor N in terms of tensor basis as

$$N_{ijkl} = \frac{a^3}{V}(n_1 T^1_{ijkl} + n_2 T^2_{ijkl}) \tag{2.41}$$

with coefficients

$$\begin{aligned} n_1 &= \frac{16}{3} \frac{\sqrt{A_2A_3}(\sqrt{A_2}+\sqrt{A_3})(2C^0_1 + C^0_2)}{(\sqrt{A_2A_3}C^0_5 + 4C^0_6)} \\ n_2 &= \frac{32}{3}\left[\sqrt{\frac{C^0_5}{2C^0_2}} + \frac{\sqrt{A_2A_3}C^0_5 + 4C^0_6}{2\sqrt{A_2A_3}(\sqrt{A_2}+\sqrt{A_3})(2C^0_1 + C^0_2)}\right]^{-1} \end{aligned} \tag{2.42}$$

and A_1, A_2 and A_3 are given by (2.18) and (2.19). If, instead of rigid inclusions, we have a crack of radius a, then the compliance contribution tensor has to be considered:

$$H_{ijkl} = \frac{a^3}{V}(h_5 T^5_{ijkl} + h_6 T^6_{ijkl}) \tag{2.43}$$

with coefficients

$$\begin{aligned} h_5 &= \frac{64}{3\sqrt{2}C^0_5}\left[\sqrt{\frac{C^0_2}{C^0_5}} + \frac{-4(C^0_3)^2 + 2C^0_6(2C^0_1 + C^0_2)}{C^0_5(\sqrt{A_2}+\sqrt{A_3})\sqrt{C^0_6(2C^0_1 + C^0_2)}}\right]^{-1} \\ h_6 &= \frac{8}{3}\frac{(\sqrt{A_2}+\sqrt{A_3})(2C^0_1 + C^0_2)}{C^0_6(2C^0_1 + C^0_2) - 2(C^0_3)^2} \end{aligned} \tag{2.44}$$

2.2.2. Strongly prolate spheroidal inclusion

In this case, $a_3 \to \infty$ and the aspect ratio $\xi \to 0$. Then, the expressions for the shape factors (2.30) are reduced to the following simple ones:

$$J^{(l)}_1 = 2, \quad \xi^2 J^{(l)}_2 = 0 \tag{2.45}$$

Using (2.18), the following expressions can be obtained

$$\begin{aligned} &\sum_{l=1}^{3} b_l = \frac{1}{2\pi C^0_2}, \quad \sum_{l=1}^{3} c_l = 0 \\ &\sum_{l=1}^{3} d_l = \frac{1}{\pi C^0_5}, \quad \sum_{l=1}^{3}(b_l - A_l a_l) = \frac{1}{2\pi(2C^0_1 + C^0_2)} \end{aligned} \tag{2.46}$$

Substitution of (2.45) and (2.46) into (2.29) leads to the expression of tensor P_{ijkl} in terms of tensor basis with coefficients

$$P_1 = \frac{1}{2(2C_1^0 + C_2^0)}, \quad P_2 = \frac{1}{2(2C_1^0 + C_2^0)} + \frac{1}{2C_2^0}, \quad P_3 = 0 \quad P_4 = 0, \quad P_5 = \frac{2}{C_5^0}, \quad P_6 = 0 \tag{2.47}$$

and therefore, formula for the Eshelby's tensor (2.31) can be written as

$$S^{\mathrm{E}}_{\mathrm{Cylinder}} = \frac{C_1^0}{2C_1^0 + C_2^0} T^{(1)} + \frac{C_1^0 + C_2^0}{2C_1^0 + C_2^0} T^{(2)} + \frac{C_3^0}{2C_1^0 + C_2^0} T^{(3)} + T^{(5)} \tag{2.48}$$

Now formulas (2.35) allow us to write expressions for property contribution tensors of a rigid cylinder and a cylindrical pore. In the case of a rigid cylinder, the stiffness contribution tensor $N_{ijkl} = \frac{V^*}{V}\sum_\alpha n_\alpha T^{(\alpha)}_{ijkl}$ has the following coefficients n_α:

$$\begin{aligned} &n_1 = \left(C_1^0 + \frac{C_2^0}{2}\right), \quad n_2 = \frac{C_2^0(2C_1^0 + C_2^0)}{C_1^0 + C_2^0}, \quad n_3 = \left(C_3^0 + \frac{C_5^0}{4}\right) \quad n_4 = \left(C_3^0 + \frac{C_5^0}{4}\right), \\ &n_5 = 2C_5^0, \quad n_6 = \infty \end{aligned} \tag{2.49}$$

For the cylindrical pore, we can write $H_{ijkl} = \frac{V^*}{V}\sum_\alpha h_\alpha T^{(\alpha)}_{ijkl}$ with the following coefficients:

$$\begin{aligned} &h_1 = \frac{2C_3^0C_4^0 - C_6^0(2C_1^0 + C_2^0)}{4C_2^0(C_3^0C_4^0 - C_1^0C_6^0)}, \quad h_2 = \left(\frac{1}{C_1^0} + \frac{2}{C_2^0}\right), \quad h_3 = \frac{C_3^0}{2(C_3^0C_4^0 - C_1^0C_6^0)}, \\ &h_4 = \frac{C_3^0}{2(C_3^0C_4^0 - C_1^0C_6^0)}, \quad h_5 = \frac{8}{C_5^0}, \quad h_6 = \frac{C_1^0}{C_1^0C_6^0 - C_3^0C_4^0} \end{aligned} \tag{2.50}$$

2.2.3. Spherical inhomogeneity

The case of spherical shape of the inhomogeneity ($a_1 = a_2 = a_3 = a$ or $\xi = 1$) loses its simplicity if the matrix does not possess isotropic properties and the formulas for components of tensors P and S^{E} are rather lengthy. First of all, the expressions for the shape factors (2.30) can be written as

$$\begin{aligned} J_1^{(l)} &= \frac{2}{1 - A_l}\left[1 - \frac{1}{2}\frac{A_l}{\sqrt{1 - A_l}}\ln\left(\frac{1 + \sqrt{1 - A_l}}{1 - \sqrt{1 - A_l}}\right)\right] \\ J_2^{(l)} &= \frac{2}{1 - A_l}\left[-1 + \frac{1}{2}\frac{1}{\sqrt{1 - A_l}}\ln\left(\frac{1 + \sqrt{1 - A_l}}{1 - \sqrt{1 - A_l}}\right)\right] \end{aligned} \tag{2.51}$$

Substitution of these results into (2.29) gives the following expressions for coefficients P_i:

$$\begin{aligned} P_1 &= \frac{1}{8\Delta_p}\left(-C_6^0 f_1 + \frac{C_5^0}{4} g_1\right) \\ P_2 &= \frac{J_1^{(1)}}{4C_2^0} + P_1 \\ P_3 &= \frac{1}{8\Delta_p}\left(C_3^0 + \frac{C_5^0}{4}\right)(f_1 - g_2) \\ P_4 &= \frac{1}{8\Delta_p}\left(C_3^0 + \frac{C_5^0}{4}\right)(f_1 - g_2) \\ P_5 &= 2P_3 + \frac{J_2^{(1)}}{2C_2^0} - \frac{1}{4\Delta_p}\left[\frac{C_5^0}{4}(f_1 - g_2) + C_6^0 f_2 - \left(C_1^0 + \frac{C_2^0}{2}\right)g_1\right] \\ P_6 &= \frac{1}{2\Delta_p}\left(-\frac{C_5^0}{4} f_1 + \left(C_1^0 + \frac{C_2^0}{2}\right)g_2\right) \end{aligned} \tag{2.52}$$

where

$$\begin{aligned} f_1 &= J_1^{(2)} - J_1^{(3)}, \quad f_2 = J_2^{(2)} - J_2^{(3)} \\ g_1 &= A_2 J_1^{(2)} - A_3 J_1^{(3)}, \quad g_2 = A_2 J_2^{(2)} - A_3 J_2^{(3)} \\ \Delta_p &= C_{11}^0 C_{44}^0 (A_2 - A_3) \end{aligned} \tag{2.53}$$

and coefficients entering expression (2.33) for Eshelby's tensor are

$$\begin{aligned} S_{S1}^{\mathrm{E}} &= \frac{2C_1^0(-4C_6^0 f_1 + C_5^0 g_1) + C_4^0(4C_3^0 + C_5^0)(f_1 - g_2)}{32\Delta_p} \\ S_{S2}^{\mathrm{E}} &= \frac{C_2^0(-4C_6^0 f_1 + C_5^0 g_1)}{32\Delta_p} + \frac{J_1^{(1)}}{4} \\ S_{S3}^{\mathrm{E}} &= \frac{C_5^0 C_6^0 (f_1 - g_2) + C_3^0[2C_5^0 g_1 - 4C_6^0(f_1 + g_2)]}{32\Delta_p} \\ S_{S4}^{\mathrm{E}} &= \frac{C_1^0(4C_3^0 + C_5^0)(f_1 - g_2) + 2C_4^0[-C_5^0 f_1 + 2g_2(2C_1^0 + C_2^0)]}{16\Delta_p} \\ S_{S5}^{\mathrm{E}} &= \frac{C_5^0\left\{4\Delta_p J_2^{(1)} + C_2^0[-2C_6^0 f_2 + (2C_1^0 + C_2^0)g_1 + 2C_3^0(f_1 - g_2)]\right\}}{16C_2^0\Delta_p} \\ S_{S6}^{\mathrm{E}} &= \frac{-2C_5^0 C_6^0 f_1 + C_3^0(4C_3^0 + C_5^0)(f_1 - g_2) + 4C_6^0 g_2(2C_1^0 + C_2^0)}{16\Delta_p} \end{aligned} \tag{2.54}$$

The property contribution tensor of a spherical inclusion of volume V^* can now be obtained via formulas (2.35). For a rigid spherical inclusion, we have the stiffness contribution tensor $N_{ijkl} = \frac{V^*}{V}\sum_\alpha n_\alpha T_{ijkl}^{(\alpha)}$ with the following coefficients:

$$\begin{aligned} n_1 &= \frac{-C_5^0 f_1 + 2g_2(2C_1^0 + C_2^0)}{16\Delta_p\Delta_n} \\ n_2 &= \frac{32C_2^0\Delta_p}{C_2^0(-4C_6^0 f_1 + C_5^0 g_1) + 8J_1^{(1)}\Delta_p} \\ n_3 &= n_4 = -\frac{(4C_3^0 + C_5^0)(f_1 - g_2)}{32\Delta_p\Delta_n} \\ n_5 &= \frac{32C_2^0\Delta_p}{4J_2^{(1)}\Delta_p + C_2^0[-2C_6^0 f_2 + (2C_1^0 + C_2^0)g_1 + 2C_3^0(f_1 - g_2)]} \\ n_6 &= \frac{-4C_6^0 f_1 + C_5^0 g_1}{16\Delta_p\Delta_n} \end{aligned} \tag{2.55}$$

where $\Delta_n = \frac{-(4C_3^0+C_5^0)^2(f_1-g_2)^2 + 4(-4C_6^0 f_1 + C_5^0 g_1)[-C_5^0 f_1 + 2g_2(2C_1^0 + C_2^0)]}{512\Delta_p^2}$.

For a spherical pore, one can write the compliance contribution tensor $H_{ijkl} = \frac{V^*}{V}\sum_\alpha h_\alpha T_{ijkl}^{(\alpha)}$ with coefficients h_α as follows:

$$\begin{aligned} &h_1 = \frac{q_6}{4(q_1 q_6 - q_3 q_4)}, \quad h_2 = \frac{1}{q_2}, \quad h_3 = -\frac{q_3}{2(q_1 q_6 - q_3 q_4)}, \quad h_4 = -\frac{q_4}{2(q_1 q_6 - q_3 q_4)}, \\ &h_5 = \frac{4}{q_5}, \quad h_6 = \frac{q_1}{q_1 q_6 - q_3 q_4} \end{aligned} \tag{2.56}$$

where

$$
\begin{aligned}
q_1 &= \frac{1}{16\Delta_p}\Big\{2(C_1^0)^2(4C_6^0 f_1 - C_5^0 g_1) + 2C_3^0 C_4^0(C_5^0 f_1 - 2C_2^0 g_2) + C_1^0[16\Delta_p - (C_3^0 + C_4^0)(4C_3^0 + C_5^0)f_1 \\
&\quad + g_2(4C_3^0(C_3^0 - C_4^0) + (C_3^0 + C_4^0)C_5^0)]\Big\} \\
q_2 &= C_2^0\left[1 + \frac{C_2^0(4C_6^0 f_1 - C_5^0 g_1)}{32\Delta_p} - \frac{J_1^{(1)}}{4}\right] \\
q_3 &= \frac{1}{16\Delta_p}\Big\{[4(C_3^0)^3 + (C_3^0)^2 C_5^0 + C_1^0 C_5^0 C_6^0](-f_1 + g_2) + 2C_3^0[8\Delta_p + 2C_1^0 C_6^0 f_1 + C_5^0 C_6^0 f_1 - C_1^0 C_5^0 g_1 \\
&\quad - 2(C_1^0 + C_2^0)C_6^0 g_2]\Big\} \\
q_4 &= \frac{1}{16\Delta_p}\Big\{[(C_4^0)^2 + C_1^0 C_6^0](4C_3^0 + C_5^0)(-f_1 + g_2) + 2C_4^0[8\Delta_p + C_1^0(4C_6^0(f_1 - g_2) - C_5^0 g_1) \\
&\quad + C_6^0(C_5^0 f_1 - 2C_2^0 g_2)]\Big\} \\
q_5 &= C_5^0 - \frac{(C_5^0)^2\left[4\Delta_p J_2^{(1)} + C_2^0(-2C_6^0 f_2 + (2C_1^0 + C_2^0)g_1 + 2C_3^0(f_1 - g_2))\right]}{32C_2^0\Delta_p} \\
q_6 &= \frac{1}{16\Delta_p}\Big\{-2C_3^0 C_4^0 C_5^0 g_1 + 2(C_6^0)^2[C_5^0 f_1 - 2g_2(2C_1^0 + C_2^0)] + C_6^0[16\Delta_p - f_1(4C_3^0(C_3^0 - C_4^0) \\
&\quad + C_5^0(C_3^0 + C_4^0)) + (C_3^0 + C_4^0)(4C_3^0 + C_5^0)g_2]\Big\}
\end{aligned}
\tag{2.57}
$$

In the limiting case of isotropic material, the formulas for compliance contribution tensor recover the corresponding expressions derived by Kachanov et al. (1994) (cracks and cylindrical pores) and by Nemat-Nasser and Hori (1993) (spherical cavities). In the case of rigid inclusions the formulas of Sevostianov and Kachanov (1999) are recovered (to within two misprints in the last paper).

3. Transversely isotropic material containing multiple inclusion

3.1. Non-interaction approximation

In this approximation, the interaction between any two inclusions is neglected and therefore, each inclusion is assumed to be loaded by the same remotely applied stress. The total inclusion compliance and stiffness tensors are taken as a sum of individual compliance tensors. If H^{NI} and N^{NI} are the compliance and stiffness tensors of non-interaction approximation, then

$$
\begin{aligned}
H_{ijkl}^{\mathrm{NI}} &= \sum H_{ijkl}, \quad S_{ijkl}^{\mathrm{eff}} = S_{ijkl}^0 + H_{ijkl}^{\mathrm{NI}} \\
N_{ijkl}^{\mathrm{NI}} &= \sum N_{ijkl}, \quad C_{ijkl}^{\mathrm{eff}} = C_{ijkl}^0 + H_{ijkl}^{\mathrm{NI}}
\end{aligned}
\tag{3.1}
$$

where S_{ijkl}^0 and C_{ijkl}^0 are compliance and stiffness tensors of the matrix. S_{ijkl}^{eff} and C_{ijkl}^{eff} are the effective compliance and effective stiffness tensors under the assumption of non-interaction approximation.

This approximation is considered to be the most important one because of several reasons. First, it identifies the proper parameters of inclusion concentration (Kachanov, 1994) and the overall anisotropy for inhomogeneity of various shapes. Second, it serves as a basis for the effective medium theories that account for interactions by placing non-interacting defects into some effective environment and the last reason is

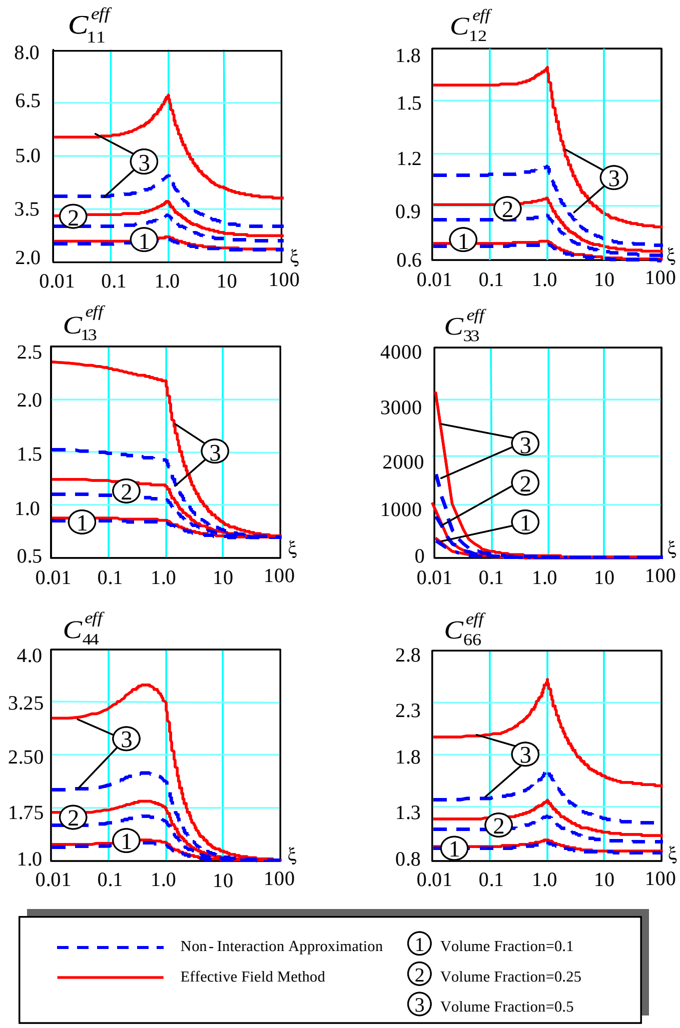

Fig. 2. Effective elastic stiffnesses of a transversely isotropic material containing parallel spheroidal rigid inclusions of identical aspect ratio ξ at various volume fractions of inhomogeneities (as functions of ξ). Comparison of calculations with non-interaction approximation (3.1) and effective field method (3.26).

that it is reasonably accurate at low inclusion volume fractions, but, for crack like inclusions, it remains accurate up to relatively high crack densities. Dashed lines in Figs. 2 and 3 illustrate the effective elastic moduli of a transversely isotropic material containing rigid inclusions and pores (calculated in the framework of non-interaction approximation) in dependence on their shapes.

3.2. Effective field method of calculation of the effective properties

To take into account the interaction of inclusions, we first consider an infinite body containing a random set of spheroidal inclusions having the same shape and orientation. As before, we denote by $V(x)$ the characteristic function of region V, occupied by the inclusions. The strain tensor $\varepsilon_{ij}(x)$ and stress tensor $\sigma_{ij}(x)$ in the composite satisfy the following relationship (Kunin, 1983):

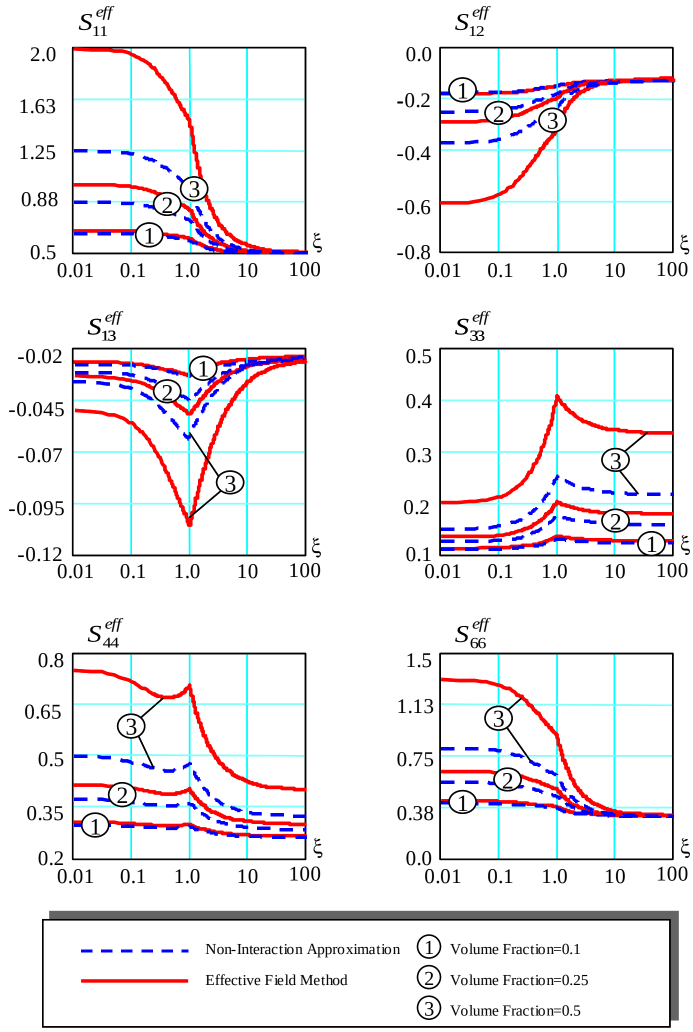

Fig. 3. Effective elastic compliances of a transversely isotropic material containing parallel spheroidal pores of identical aspect ratio ξ at various volume fractions of inhomogeneities (as functions of ξ). Comparison of calculations with non-interaction approximation (3.1) and effective field method (3.26) (inverted for compliances).

$$\varepsilon_{ij}(x) = \varepsilon_{ij}^0(x) + \int P_{ijkl}(x - x')q_{kl}(x')\mathrm{d}x' \tag{3.2}$$

$$\sigma_{ij}(x) = \sigma_{ij}^0(x) + \int Q_{ijkl}(x - x')S_{klmn}^0 q_{mn}(x')\mathrm{d}x' \tag{3.3}$$

where

$$q_{ij}(x) = C_{ijkl}^1 \varepsilon_{kl}(x)V(x) \tag{3.4}$$

Here ε^0 and σ^0 are the external strain and stress fields acting on the medium, the kernels $P(x)$ and $Q(x)$ are determined in (2.7). The integration in (3.2) and (3.3) is over the entire space. On the basis of (3.2) and (3.3), we can write the expressions for the mean values of the strain and stress fields in the form

$$\begin{aligned}\langle\varepsilon_{ij}(x)\rangle &= \varepsilon^0_{ij} + \int P_{ijkl}(x-x')\langle q_{kl}(x')\rangle \mathrm{d}x' \\ \langle\sigma_{ij}(x)\rangle &= \sigma^0_{ij} + \int Q_{ijkl}(x-x')S^0_{klmn}\langle q_{mn}(x')\rangle \mathrm{d}x'\end{aligned} \tag{3.5}$$

For a spacial uniform set of inclusions $\varepsilon_{ij}(x)$, $\sigma_{ij}(x)$ and $q_{ij}(x)$ are homogeneous random ergodic functions. Hence, $\langle q\rangle$ is a constant tensor whose value can be found by spacial averaging of a typical fixed realization of the random function $q(x)$. Because of the linearity of the problem the strain field $\varepsilon_{ij}(x)$ is represented by the external field ε^0_{ij} through the relation

$$\varepsilon_{ij}(x) = \Lambda_{ijkl}(x)\varepsilon^0_{kl} \tag{3.6}$$

where $\Lambda(x)$ is a certain random function of coordinates. This function has to be obtained from the solution of many-particles problem. After substituting the expression for $\varepsilon(x)$ in the formula for $q(x)$ and averaging the result, we obtain

$$\langle q_{ij}\rangle = pC^{\Lambda}_{ijkl}\varepsilon^0_{kl}, \quad C^{\Lambda}_{ijkl} = C^1_{ijmn}\langle\Lambda_{mnkl}\rangle \tag{3.7}$$

where p is the volume concentration of the inclusions ($p = \langle V(x)\rangle$).

It is assumed henceforth that the average strain in the composite $\langle\varepsilon_{ij}\rangle$coincides with the external field ε^0_{ij} and does not depend on the properties and concentration of the inclusions ($\langle\varepsilon_{ij}\rangle = \varepsilon^0_{ij}$). This mean value is determined by the conditions at infinity. In this case, the question arises of the action of integral operators with kernels $P(x)$ and $Q(x)$ on constants. It was shown in Kanaun (1983) that the unique definition of these actions depends on a given type of external field (specified in the problem): the stress field σ^0_{ij} or strain field ε^0_{ij}. For instance, if the stress fields is fixed, operator P and Q act on constants as follows:

$$\int Q(x-x')\mathrm{d}x' = 0, \quad \int P(x-x')\mathrm{d}x' = (S^0)^{-1} \tag{3.8}$$

However, if the deformation of the medium is constrained at infinity (the strain tensor is fixed as was supposed above), the result will be different

$$\int Q(x-x')\mathrm{d}x' = -C^0, \quad \int P(x-x')\mathrm{d}x' = 0 \tag{3.9}$$

Hence, we can write in this case

$$\langle\sigma_{ij}\rangle = C^{\mathrm{eff}}_{ijkl}\langle\varepsilon_{kl}\rangle, \quad C^{\mathrm{eff}}_{ijkl} = C^0_{ijkl} + pC^{\Lambda}_{ijkl} \tag{3.10}$$

where C^{eff} is the tensor of the effective elastic moduli of the composite.

Thus, the problem of obtaining the effective elastic moduli C^{eff} reduces to evaluation of the tensor C^{Λ} determined in Eq. (3.7). This tensor depends on the solution of many-particle problem through the function $\Lambda(x)$. For evaluation of C^{Λ}, we use below the self-consistent scheme named effective field method (EFM). This method has a long history and was mainly used in the nuclear physics and in the theory of phase transitions for description of many-particle interaction. In application to the mechanics of composite materials, this method was developed by Kanaun (1983), Kanaun and Levin (1993, 1994), Markov (1999).

Let us consider an arbitrary ith inclusion that occupies the region V_i in a fixed typical realization of random set of inhomogeneities. We denote by $\varepsilon^*_{ij(k)}(x)$ the local external field acting on this inclusion. The field $\varepsilon^*_{ij(k)}(x)$ is composed of the external field ε^0 and the disturbances of the field due to surrounding inclusions. Self-consistent schemes in which interaction between inclusions are taken into account by introducing local external field acting on each inclusions are called the effective field method. Let us introduce the field $\varepsilon^*_{ij}(x)$ that coincides with $\varepsilon^*_{ij(k)}(x)$ inside the region V_k. It follows from (3.2) that

$$\varepsilon^*_{ij}(x) = \varepsilon^0_{ij} + \int P_{ijkl}(x - x')C^1_{klmn}\varepsilon_{mn}(x')V(x;x')\mathrm{d}x' \tag{3.11}$$

where $V(x;x')$ is the characteristic function (with argument x') of the region V_x, defined by the relation

$$V_x = \bigcup_{i \neq k} V_i \quad \text{when} \quad x \in V_k \tag{3.12}$$

In the simplest variant of the effective field method, let us introduce two simplifying assumptions concerning the structure of the field $\varepsilon^*_{ij}(x)$, i.e. hypothesis of the EFM:

1. The field $\varepsilon^*_{ij}(x)$ is constant (uniform) in each region occupied by inclusions and is the same for all inclusions.
2. The random field $\varepsilon^*_{ij(k)}(x)$ does not depend statistically on the geometrical characteristics and elastic properties of the kth inclusion, occupying region V_k.

Using hypothesis one, we obtain that the strain field $\varepsilon_{ij}(x)$ is connected to ε^*_{ij} by the relation obtained above for the single inclusion

$$\varepsilon_{ij} = A_{ijkl}\varepsilon^*_{kl}, \quad A = (I + PC^1)^{-1} \tag{3.13}$$

Note that for the spheroidal inclusions of the same shapes and orientation A is the constant tensor that is the same for all inclusions.

Relation (3.13) allows to express the strain field $\varepsilon(x)$ in the arbitrary point x of the composite via the local effective field ε^*

$$\varepsilon_{ij}(x) = \varepsilon^0_{ij} + \int P_{ijkl}(x - x')C^A_{klmn}\varepsilon^*_{mn}V(x;x')\mathrm{d}x', \tag{3.14}$$

$$C^A_{ijkl} = C^1_{ijmn}A_{mnkl}$$

and to obtain the self-consistent equation for field ε^* determination

$$\varepsilon^*_{ij}(x) = \varepsilon^0_{ij} + \int P_{ijkl}(x - x')C^A_{klmn}\varepsilon^*_{mn}V(x;x')\mathrm{d}x' \tag{3.15}$$

Let us average the both sides of Eq. (3.15) under the condition that $x \in V$. Taking into account hypothesis 2 of the EFM, we can write

$$\langle \varepsilon^*_{ij}(x) \mid x \rangle = \varepsilon^0_{ij} + p\int P_{ijkl}(x - x')C^A_{klmn}\varepsilon^*_{mn}\Psi(x - x')\mathrm{d}x' \tag{3.16}$$

$$\Psi(x - x') = \frac{\langle V(x;x') \mid x \rangle}{\langle V(x) \rangle}$$

Here symbol $\langle \cdot \mid x \rangle$means the ensemble average provided that point x is in the region V. It follows from definition (3.12) for $V(x;x')$ that $\Psi(x)$ is a continuous function and

$$\Psi(x) = 0 \quad \text{when} \quad x = 0 \tag{3.17}$$

Because of the weakening in geometrical linkage between the positions of the inclusions when the distance between them increase, the following relation takes place

$$\Psi(x) \to 1 \quad \text{when} \quad \mid x \mid \to \infty \tag{3.18}$$

Function $\Psi(x)$ defines the shape of the "correlation hole" inside of which a typical inclusion is located (the region in the vicinity of each inclusion the finding in which the center of some other inclusion is improbable). If the random set of inclusions possesses some symmetry (in the statistical sense) it influences the symmetry of function $\Psi(x)$. In the case when the random set of the inclusions is statistically isotropic, function $\Psi(x)$ is spherically symmetric, i.e. $\Psi(x) = \Psi(|x|)$.

The deviation from the isotropic distribution of the random set of inclusions can lead to texture. In many cases, such texture can be describe by a two-rank tensor α_{ij}. This tensor determines the linear space transform, which converts function $\Psi(x)$ into a spherically symmetric one

$$y_i = \alpha_{ij}x_j, \quad \Psi(\alpha^{-1}y) = \Psi(|y|) \tag{3.19}$$

In this case, the ellipsoid A, defined by the expression $|\alpha x| \leqslant 1$ with semi-axes α_1, α_2 and α_3 describes the shape of the correlation hole. Taking into account the relations (3.9) when the strain field is fixed in the problem, we can write Eq. (3.16) in the form

$$\begin{aligned}&\langle \varepsilon^*_{ij}(x) \mid x\rangle = \varepsilon^0_{ij} - p\int P_{ijkl}(x-x')C^A_{klmn}\varepsilon^*_{mn}\Phi(x-x')\mathrm{d}x'\\&\Phi(x) = 1-\Psi(x)\end{aligned} \tag{3.20}$$

Identifying now the mean $\langle \varepsilon^*_{ij}(x) \mid x\rangle$ with the effective field ε^*_{ij}, we can find from (3.20)

$$\varepsilon^*_{ij} = D_{ijkl}\langle\varepsilon_{kl}\rangle, \quad D = (I - pP^{\Phi}C^A)^{-1} \tag{3.21}$$

where it is denoted

$$P^{\Phi}_{ijkl} = -\int P_{ijkl}(x)\Phi(x)\mathrm{d}x \tag{3.22}$$

If we assume in the considered case that the shape of the correlation hole id spheroidal (concentric with inclusion but not necessary with the same aspect ratio) then tensor P^{Φ} can be calculated explicitly.

After the connection (3.21) is established, it can be substituted in (3.13) and then we can find according to (3.7)

$$C^A = C^A(I - pP^{\Phi}C^A)^{-1} \tag{3.23}$$

and

$$C^{\mathrm{eff}}_{ijkl} = C^0_{ijkl} + pC^A(I - pP^{\Phi}C^A)^{-1} \tag{3.24}$$

In the special case, the shape of the correlation hole can be coincided with the shape of inclusion itself. In this case, $P^{\Phi} = P$ and expression (3.24) is simplified

$$C^{\mathrm{eff}}_{ijkl} = C^0_{ijkl} + p[(C^1_{ijkl})^{-1}(1-p)P_{ijkl}]^{-1} \tag{3.25}$$

Note that the same expression for the tensor of effective elastic moduli can be obtained for by so-called Mori–Tanaka method (Mori and Tanaka, 1973), (Benveniste, 1987). This method is based on the assumption that every inclusion in material behaves as isolated one in the matrix and undergoes a constant external field that is assumed to coincided with the average strain field in the matrix. Hence, Mori–Tanaka's method gives the result coincided with the effective field method if we accept as an additional assumption that the shape of correlation hole coincides with the shape of typical inclusion. In general, these shapes can be different and tensors P^{Φ} and P are not the same.

As it follows from (3.25), the composite material is macroscopically transversely isotropic and is characterized by five independent effective elastic moduli. In T-basis tensor, C^{eff} can be written as

$$C^{\mathrm{eff}} = C^{\mathrm{eff}}_1T^1 + C^{\mathrm{eff}}_2T^2 + C^{\mathrm{eff}}_3T^3 + C^{\mathrm{eff}}_4T^4 + C^{\mathrm{eff}}_5T^5 + C^{\mathrm{eff}}_6T^6 \tag{3.26}$$

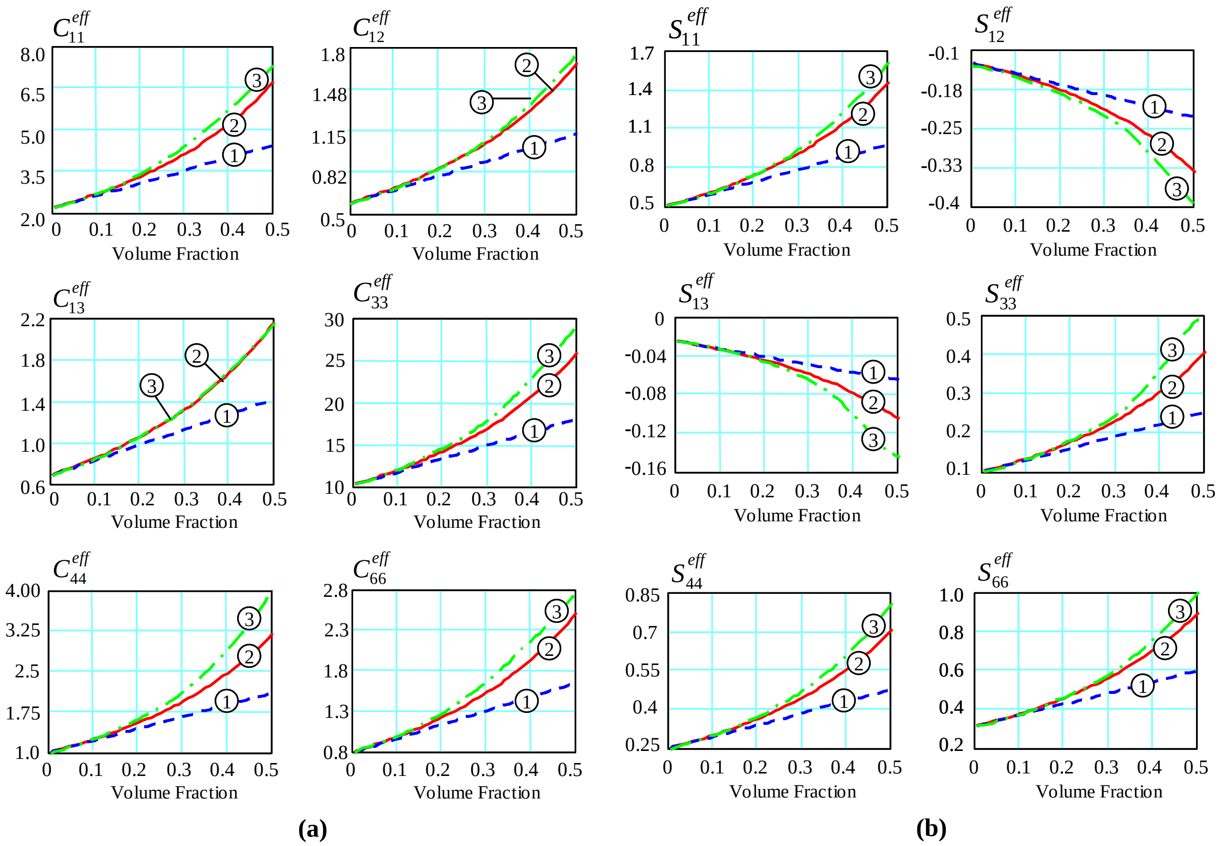

Fig. 4. Effective elastic constants of a transversely isotropic material containing randomly located rigid spherical inclusions. Comparison of non-interaction, effective field and quasi-random lattice methods: (a) components of the elastic stiffness tensor and (b) engineering constants.

where

$$\begin{aligned}
C_1^{\text{eff}} &= C_1^0 + \frac{p}{2\Delta^*}\left[\frac{2C_1^1}{\Delta_1} + (1-p)P_6\right] \\
C_2^{\text{eff}} &= C_2^0 + p\left[\frac{1}{C_2^1} + (1-p)P_2\right]^{-1} \\
C_3^{\text{eff}} &= C_3^0 + \frac{p}{\Delta^*}\left[\frac{C_3^1}{\Delta_1} - (1-p)P_3\right] \\
C_4^{\text{eff}} &= C_3^{\text{eff}} \\
C_5^{\text{eff}} &= C_5^0 + 4p\left[\frac{4}{C_5^1} + (1-p)P_5\right]^{-1} \\
C_6^{\text{eff}} &= C_6^0 + \frac{p}{\Delta^*}\left[\frac{C_6^1}{\Delta_1} + 2(1-p)P_1\right]
\end{aligned} \tag{3.27}$$

$$\Delta_1 = C_1^1 C_6^1 - (C_3^1)^2, \quad \Delta^* = 2\left\{\left[\frac{C_6^1}{2\Delta_1} + (1-p)P_1\right]\left[\frac{2C_1^1}{\Delta_1} + (1-p)P_6\right] - \left[\frac{C_3^1}{\Delta_1} - (1-p)P_3\right]^2\right\} \tag{3.28}$$

3.3. Comparison with the "unit cell" method

In the case of spherical inhomogeneities, we also compared our results with the calculations done via the "unit cell" method (Kushch and Sevostianov, 2004). The basic idea of this method consists in modeling an actual micro geometry of composite by the idealized periodic structure with a unit cell containing from one to several particles, for which the homogenization boundary-value problem is to be stated and solved. Sometimes, this model is referred also as the "lattice" model reflecting the fact that the centers of inclusions form a spatially periodic lattice. This model provides a natural way, through the periodic boundary conditions on the opposite cell facets, to take into account interactions among whole infinite array of inhomogeneities. Also, the deterministic structure of unit cell enables an accurate solution of the corresponding periodic boundary-value problems. These features make the unit cell approach one of the most appropriate numerical methods for studying the high-filled strongly heterogeneous composites, where the structure and interactions between the particles should be taken into account to a maximum possible extent.

Figs. 4 and 5 illustrate comparison of effective elastic constants calculated via non-interaction approximation and effective field method of Levin and Kanaun with those obtained in the paper of Kushch and Sevostianov (2004) by quasi-random variant of the "unit cell" method.

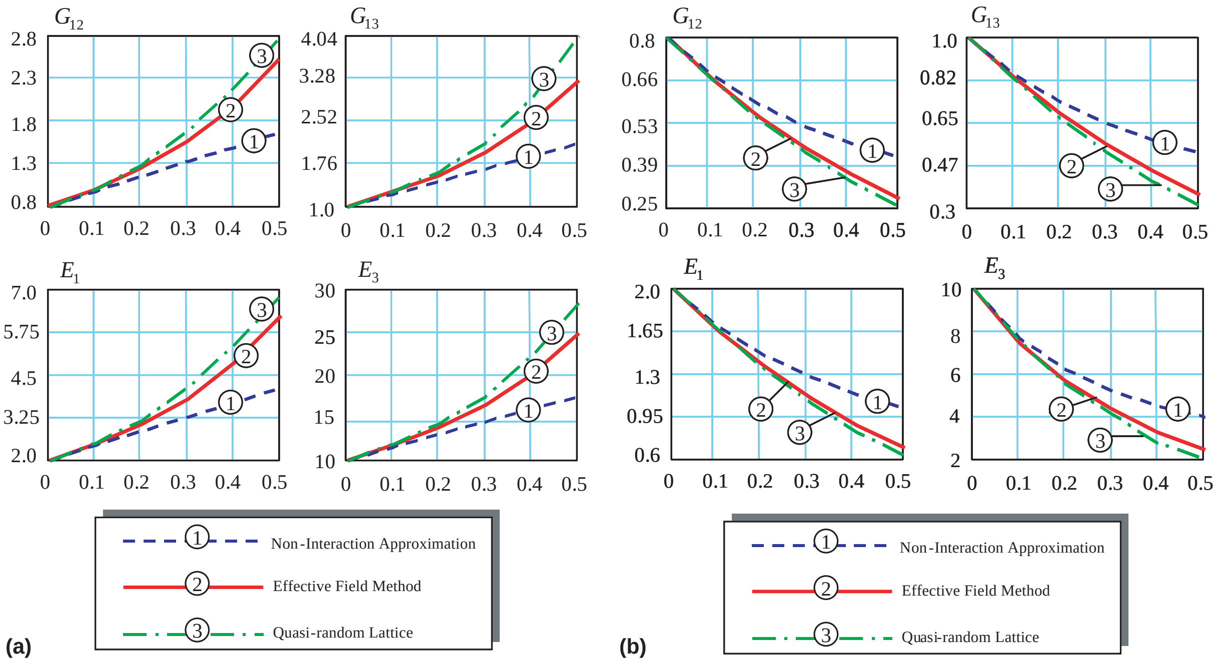

Fig. 5. Effective elastic constants of a transversely isotropic material containing randomly located spherical pores. Comparison of non-interaction, effective field and quasi-random lattice methods: (a) components of the elastic compliance tensor and (b) engineering constants.

4. Conclusions

The paper addresses the problem of calculation of the effective elastic properties of an inhomogeneous material consisting of transversely isotropic phases. As a first step, the problem about a transversely isotropic medium containing a single inhomogeneity is solved. Eshelby's tensor for such a problem is rederived in the form suitable for calculation of the effective properties. Then, the stiffness/compliance contribution tensors for such an inclusion are constructed. Applying the methodology of Kachanov et al. (1994), the property contribution tensors are used for approximation of the effective elastic properties of composites. In the present paper, the derivation is done in the frameworks of non-interaction approximation and the effective field method of Kanaun (1983), Kanaun and Levin (1993, 1994), Markov (1999). The results are compared with numerical simulations via the "unit cell" method given by Kushch and Sevostianov (2004). Application of other approximate schemes (Mori–Tanaka's scheme, differential method, effective media scheme) may be done via general formulas given in the paper of Tsukrov and Eroshkin (2004) on the base of non-interaction approximation.

Appendix A. Tensorial basis in the space of transversely isotropic fourth rank tensors: representation of certain transversely isotropic tensors in terms of the tensorial basis

The operations of analytic inversion and multiplication of fourth rank tensors are conveniently done in terms of special tensorial bases that are formed by combinations of unit tensor and one or two orthogonal unit vectors (see Kunin, 1983 and Kanaun and Levin, 1993). In the case of the transversely isotropic elastic symmetry, the following basis is most convenient (it differs slightly from the one used by Kanaun and Levin, 1993):

$$T^{(1)}_{ijkl} = \Theta_{ij}\Theta_{kl}, \quad T^{(2)}_{ijkl} = (\Theta_{ik}\Theta_{lj} + \Theta_{il}\Theta_{kj} - \Theta_{ij}\Theta_{kl})/2, \quad T^{(3)}_{ijkl} = \Theta_{ij}m_k m_l, \quad T^{(4)}_{ijkl} = \Theta_{ij}m_k m_l,$$
$$T^{(5)}_{ijkl} = (\Theta_{ik}m_l m_j + \Theta_{il}m_k m_j + \Theta_{jk}m_l m_i + \Theta_{jl}m_k m_i)/4, \quad T^{(6)}_{ijkl} = m_i m_j m_k m_l, \tag{A.1}$$

where $\Theta_{ij} = \delta_{ij} - m_i m_j$ and $m = m_1 e_1 + m_2 e_2 + m_3 e_3$ is a unit vector along the axis of transverse symmetry. These tensors form a closed algebra with respect to the operation of (non-commutative) multiplication (contraction over two indices):

$$(T^{(\alpha)} : T^{(\beta)})_{ijkl} \equiv T^{(\alpha)}_{ijpq} T^{(\beta)}_{pqkl} \tag{A.2}$$

The table of multiplication of these tensors has the following form (the column represents the left multiplier)

	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$	$T^{(5)}$	$T^{(6)}$
$T^{(1)}$	$2T^{(1)}$	0	$2T^{(3)}$	0	0	0
$T^{(2)}$	0	$T^{(2)}$	0	0	0	0
$T^{(3)}$	0	0	0	$T^{(1)}$	0	$T^{(3)}$
$T^{(4)}$	$2T^{(4)}$	0	$2T^{(6)}$	0	0	0
$T^{(5)}$	0	0	0	0	$T^{(5)}/2$	0
$T^{(6)}$	0	0	0	$T^{(4)}$	0	$T^{(6)}$

Then the inverse of any fourth rank tensor, as well as the product of two such tensors are readily found in the closed form, as soon as the representation in the basis

$$X = \sum_{k=1}^{6} X_k T^{(k)}, \quad Y = \sum_{k=1}^{6} Y_k T^{(k)} \tag{A.3}$$

are established. Indeed

(a) Inverse tensor X^{-1} defined by is given by $X^{-1}_{ijmn} X_{mnkl} = (X_{ijm} X^{-1}_{mnkl}) = J_{ijkl}$ is given by

$$X^{-1} = \frac{X_6}{2\Delta} T^{(1)} + \frac{1}{X_2} T^{(2)} - \frac{X_3}{\Delta} T^{(3)} - \frac{X_4}{\Delta} T^{(4)} + \frac{4}{X_5} T^{(5)} + \frac{2X_1}{\Delta} T^{(6)} \tag{A.4}$$

where $\Delta = 2(X_1 X_6 - X_3 X_4)$.

(b) Product of two tensors $X{:}Y$ (tensor with *ijkl* components equal to $X_{ijmn} Y_{mnkl}$) is

$$\begin{aligned} X : Y = {} & (2X_1 Y_1 + X_3 Y_4) T^{(1)} + X_2 Y_2 T^{(2)} + (2X_1 Y_3 + X_3 Y_6) T^{(3)} + (2X_4 Y_1 + X_6 Y_4) T^{(4)} \\ & + \frac{1}{2} X_5 Y_5 T^{(5)} + (X_6 Y_6 + 2X_4 Y_3) T^{(6)}. \end{aligned} \tag{A.5}$$

If x_3 is the axis of transverse symmetry, tensors $T^{(1)}, \ldots, T^{(6)}$ given by (A.1) have the following non-zero components:

$$\begin{aligned} & T^{(1)}_{1111} = T^{(1)}_{2222} = T^{(1)}_{1122} = T^{(1)}_{2211} = 1 \\ & T^{(2)}_{1212} = T^{(2)}_{2121} = T^{(2)}_{1221} = T^{(2)}_{2112} = T^{(2)}_{1111} = T^{(2)}_{2222} = \frac{1}{2} \\ & T^{(2)}_{1122} = T^{(2)}_{2211} = -\frac{1}{2} \\ & T^{(3)}_{1133} = T^{(3)}_{2233} = 1 \\ & T^{(4)}_{3311} = T^{(4)}_{3322} = 1 \\ & T^{(5)}_{1313} = T^{(5)}_{2323} = T^{(5)}_{1331} = T^{(5)}_{2332} = T^{(5)}_{3113} = T^{(5)}_{3223} = T^{(5)}_{3131} = T^{(5)}_{3232} = \frac{1}{4} \\ & T^{(6)}_{3333} = 1 \end{aligned} \tag{A.6}$$

A general transversely isotropic symmetric fourth rank tensor, being represented in this basis $\Psi_{ijkl} = \sum \psi_m T^m_{ijkl}$ has the following components:

$$\psi_1 = (\Psi_{1111} + \Psi_{1122})/2, \quad \psi_2 = 2\Psi_{1212}, \quad \psi_3 = \Psi_{1133}, \quad \psi_4 = \Psi_{3311}, \quad \psi_5 = 4\Psi_{1313}, \quad \psi_6 = \Psi_{3333} \tag{A.7}$$

In this manner, we could write the components of the stiffness tensor as follows:

$$C^0_1 = (C^0_{1111} + C^0_{1122})/2, \quad C^0_2 = 2C^0_{1212}, \quad C^0_3 = C^0_{1133}, \quad C^0_4 = C^0_{3311}, \quad C^0_5 = 4C^0_{1313}, \quad C^0_6 = C^0_{3333} \tag{A.8}$$

References

Benveniste, J.A., 1987. A new approach to the application of the Mori–Tanaka's theory in composite materials. Mech. Mater. 6, 147–157.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. R. Soc. London A 241, 376–396.
Eshelby, J.D., 1959. The elastic field outside an ellipsoidal inclusion. Proc. R. Soc. London A 252, 561–569.
Eshelby, J.D., 1961. Elastic inclusions and inhomogeneities. In: Hill, R., Sneddon, I.N. (Eds.), Progress in Solid Mechanics, vol. 2. North-Holland, Amsterdam, pp. 89–140.
Hashin, Z., 1983. Analysis of composite materials. J. Appl.Mech. 50, 481–505.
Kachanov, M., 1994. Solids with cracks and non-spherical pores: proper parameters of defect density and effective properties. Int J. Fracture 97, 1–32.
Kachanov, M., Sevostianov, I., 2004. On quantitative characterization of microstructures and effective properties. Int. J. Sol. Struct., this issue, doi:10.1016/j.ijsolstr.2004.06.016.
Kachanov, M., Tsukrov, I., Shafiro, B., 1994. Effective moduli of solids with cavities of various shapes. Appl. Mech. Rev. 47, S151–S174.
Kanaun, S.K., 1983. Elastic medium with random field of inhomogeneities. In: Kunin, I.A. (Ed.), Elastic Media with Microstructure, vol. 2. Springer, Berlin, pp. 165–228, Chapter 7.
Kanaun, S.K., Levin, V.M., 1993. The Effective Field Method in the Mechanics of Composite Materials. Petrozavodsk University Publishers, Petrazavodsk (in Russian).
Kanaun, S.K., Levin, V.M., 1994. Effective field method in mechanics of matrix composite materials. In: Markov, K.Z. (Ed.), Recent Advances in Mathematical Modelling of Composite Materials. World Scientific, Singapore, pp. 1–58.
Kroner, E., 1953. Das Fundamental integral der anisotropen elastischen Differentialgleichungen. Z. Phys. 136, 402–410.
Kunin, I.A., 1983. Elastic Media with Microstructure. Springer, Berlin.
Kushch, V., Sevostianov, I., 2004. Effective elastic moduli tensor of particulate composite with transversely isotropic phases. Int. J. Solid Struct. 41, 885–906.
Markov, K.Z., 1999. Elementary micromechanics of heterogeneous media. In: Markov, K.Z., Preziosi, L. (Eds.), Heterogeneous Media: Micromechanics Modeling Methods and Simulations. Birkhauser, Basel, pp. 1–162.
Mori, S., Tanaka, K., 1973. Average stress in matrix and average energy of materials with misfiting inclusions. Acta Metall. 21, 571–574.
Mura, T., 1982. Micromechanics of defects in solids. Martinus Nijhoff Publishers, The Netherlands.
Nemat-Nasser, S., Hori, M., 1993. Micromechanics: Overall Properties of Heterogeneous Materials. North-Holland, Amsterdam.
Sevostianov, I., Levin, V.M., Kachanov, M., 2001. On the modeling and design of piezocomposites with prescribed properties. Arch. Appl. Mech. 71, 673–693.
Sevostianov, I., Kachanov, M., 1999. Compliance tensors of ellipsoidal inclusions. Int. J. Fracture 96, L3–L7.
Sevostianov, I., Kachanov, M., 2002. Explicit cross-property correlations for anisotropic two-phase composite materials. J. Mech. Phys. Solids 50, 253–282.
Tsukrov, I., Eroshkin, O., submitted. On micromechanical modeling of three-dimensional solids with inclusions of various shapes Int. J. Sol. Struct., this issue, doi:10.1016/j.ijsolstr.2004.06.045.
Vakulenko, A.A., 1998. On the effective properties and strength of ceramic composites. Mech. Solids 6, 50–71.
Withers, P.J., 1989. The determination of the elastic field of an ellipsoidal inclusion in a transversely isotropic medium, and its relevance to composite materials. Philos. Mag. A-59, 759–781.
Yoo, M.N., 1974. Elastic interaction of small defects and defect clasters. Phys. Stat. Sol. B-61, 411–418.